Implications of Unequal Rates of Population Growth for Trade Revisited: Lessons from Closed Form Solutions to An Overlapping Generations General Equilibrium Model under Autarky and Trade Scenarios

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Implications of Unequal Rates of Population Growth for Trade Revisited: Lessons from Closed Form Solutions to An Overlapping Generations General Equilibrium Model under Autarky and Trade Scenarios*

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Abstract

We study closed form solutions that we obtained from a two-sector, two-factor overlapping generations model under autarky and free trade scenarios for an *á la* Heckscher-Ohlin exploration of the possible implications of population growth differentials for the patterns of trade flows between economies that are identical except for population growth rates. Our analysis shows that differences in population growth rates give way to differences in relative commodity and factor prices, creating the basis for comparative advantages in the same way as suggested by the static Heckscher-Ohlin model. We also show that unequal rates of population growth prevent comparative advantages from getting eliminated in the long-run, thereby allowing trade to continue to occur even after the steady state is reached. Our solutions reveal, however, that trade does not necessarily improve welfare for both parties in the long-run. The explanation we offer for this nicely complements previous studies that obtained similar results by using overlapping generations general equilibrium models within two country set-ups with steady populations.

Keywords: Dynamic trade; Population growth rate; Overlapping-generations general equilibrium model, Heckscher-Ohlin
JEL classification: F11, D91, J10

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1 Introduction

By the static 2x2x2 Heckscher-Ohlin (HO) model of international trade, differences in relative factor endowments across countries suffice to render trade Pareto-superior to autarky, as long as the factor intensity of production is different for each commodity. While this model has proved to be a very popular starting point for many theoretical and empirical studies, only a few studies in the literature have investigated the validity of predictions of the standard HO model in a dynamic framework. This lack of interest was possibly due to the fact that trade itself would, in the long-run, eliminate the initial differences between relative factor endowments of countries that are assumed to be identical in every other respect, thereby leaving no further incentives for partners to continue trading (Chen [2]). This paper argues that trade may continue to occur in the long-run if there are additional differences to make factor proportions evolve over time, and shows that differential speed of population growth between trading nations is one of the attributes that could lead to such an evolution in relative factor endowments, guaranteeing the continuation of trade in the long-run.

This is, in fact, an emperically relevant example to such attributes, as the United Nations projections indicate that the existing gap between the population growth rates in relatively labour-abundant nations of the developing world and relatively capital-abundant developed nations are likely to remain visible even beyond the year 2050. Even at present, labour forces in these areas continue to diverge as population pyramids get reshaped with differential paces of growth in the shares of people just reaching the working age, and of those that leave the workforce due to ageing. The resulting variations in the age profiles of nations are also affecting relative magnitudes of savings and capital accumulation, implying additional changes in relative factor endowments (Kenc and Sayan [6]).

Although changes in relative factor endowments arising due to the differential speed of demographic transition in developing and developed parts of the world are gradually becoming a major factor to affect future patterns of trade,[1] the dynamic trade literature has largely overlooked this issue so far (see Sayan [9] for a brief survey of the literature). This paper aims to contribute to the literature by extending the static HO model into a dynamic, overlapping generations set-up to look into the role that the differences in the population growth rates across nations could play as a determinant of long-run comparative advantages and to discuss the validity of welfare predictions of the static HO model in the long-run.

For this purpose, we consider a world that is made up of two countries/regions each producing two commodities by using capital and labour. We assume that countries are identical in every other respect than the rates of population growth, and study the implications of this for trade by solving the autarky and trade models analytically. The economies we consider are populated by individuals that live for two periods, and the population in each is allowed to grow constantly at a distinct rate. Such an overlapping generations (OLG) structure capturing the changing savings behaviour of individuals over the working and retirement phases of the life cycle implicitly allows the share of savings in national incomes to differ across countries, as differences in the speed of population growth induce variations in relative shares of different age groups in populations. Thus, relative factor endowments evolve, due not only to the changes in labour supply, but also to the changes in capital accumulation resulting from the changing age profiles of populations.

Investigation of the welfare implications of trade in the long-run within this set-up is particularly interesting, as a number of studies based on OLG models with stationary populations have previously suggested that trade would not necessarily lead to mutual welfare gains, and might not even be Pareto-superior to autarky in the long-run (see Mountford [7] and Sayan [8] for examples from the existing

[1]While this paper looks at the effects of differential speed of demographic transition on future patterns of trade, a recent paper by Galor and Mountford [4] reverses the question and tries to explain the historically observed effects of trade on variations in the speed of demographic transition across countries.

literature and more detailed discussions).

The organisation of the paper is as follows. The next section describes the model, presents the closed form solutions for the case of autarky, and discusses the implications of differential speed of population growth between countries. Section 3 presents the closed form solutions of the world model under the assumption of free trade. Section 4 concludes the paper by evaluating the results from autarky and trade scenarios.

2 Assumptions and Behavioural Equations of the Model

We study the long-run equilibrium under each of autarky and trade, by using an infinite horizon overlapping generations model with perfect foresight. Since the countries considered are initially assumed to be exactly the same in every respect but population growth rates, solving the autarky model for one and finding the sensitivity of the steady state value of each variable to changes in the population growth rate would be enough to compare long-run autarky solutions across countries. So, we begin by listing the assumptions that are common to both countries, and write the equations without indexing variables to countries.

Each country is assumed to be populated by individuals who live for two periods. At every period t, a generation made up of N_t individuals is born. Population grows at the constant rate n so that $N_t = (1+n)N_{t-1}$. For all periods t, individuals born and living the first period of their lives at time t inelastically supply a fixed amount of labour, earn labour income at the competitive wage rate, w_t, and decide on how to allocate it between the first period consumption of goods 1 and 2 (c_{1yt}, c_{2yt}), and savings, s_t, which bring interest earnings at the rate of r_{t+1} the next period. In the second period, they retire and consume c_{1ot+1} units of good 1, and c_{2ot+1} units of good 2 by spending all their capital income from previous period's savings.

On the supply side, two commodities are produced by using labour, L, and capital, K, under constant returns to scale, Cobb-Douglas type production technologies that are the same across countries for each commodity but different across commodities. In other words, sectoral production technologies are common to both countries but production is relatively capital-intensive in one sector, and relatively labour-intensive in the other, just as in the original formulation of static HO model.

As differently from overlapping generations general equilibrium models in such studies as Galor [3] and Azariadis [1], our formulation allows good 1 be used for consumption as well as investment purposes.[2] So, we take this good as the capital-intensive one, whereas we take good 2 that serves as a consumption good alone, as the labour-intensive one (see Sayan [8]).

Under these assumptions, consumption and production decisions and long-run autarky equilibrium in each economy can be described as follows.

2.1 Consumption and Saving

Given the price, p_t, of the consumption good (good 2) in terms of the investment-consumption good (good 1) at time t, each individual solves the following problem,

$$\begin{array}{rll} max & (c_{1yt}^{\theta}c_{2yt}^{1-\theta})^{\mu}(c_{1ot+1}^{\theta}c_{2ot+1}^{1-\theta})^{1-\mu} & \\ \text{subject to} & c_{1yt}+p_tc_{2yt}+\frac{1}{1+r_{t+1}}(c_{1ot+1}+p_{t+1}c_{2ot+1}) & = w_t\bar{l}, \\ & c_{1yt}, c_{2yt}, c_{1ot+1}, c_{2ot+1} & \geq 0, \end{array} \tag{1}$$

where $0<\theta<1$ and $0<\mu<1$.

The solution to this problem results in the following consumption decisions:

$$c_{1yt} = \mu\theta w_t\bar{l}, \tag{2}$$

$$c_{2yt} = \mu(1-\theta)\frac{w_t\bar{l}}{p_t}, \tag{3}$$

[2]While this makes the model relatively more realistic, it also adds to the complexity of the utility maximisation problem, since the consumers are now required to decide how much to consume of each good every period.

$$
\begin{aligned}
c_{1ot+1} &= (1-\mu)\theta(1+r_{t+1})w_t\bar{l}, &(4)\\
c_{2ot+1} &= (1-\mu)(1-\theta)(1+r_{t+1})\frac{w_t\bar{l}}{p_{t+1}}. &(5)
\end{aligned}
$$

implying that the private saving rate is given by $(1-\mu)$.

2.2 Production

Given the assumptions about production technologies listed above, sectoral outputs X_{1t} and X_{2t} can be expressed in per capita terms as $x_{1t} = k_{1t}^{\alpha} l_{1t}^{1-\alpha}$ and $x_{2t} = k_{2t}^{\beta} l_{2t}^{1-\beta}$ (for $0 < \alpha < 1$, $0 < \beta < 1$) where $x_{it} = \frac{X_{it}}{N_t}$, $k_{it} = \frac{K_{it}}{N_t}$ and $l_{it} = \frac{L_{it}}{N_t}$ for $i = 1, 2$. The value of parameter α is assumed to be greater than β to allow production of investment-consumption good (1) to be relatively more capital-intensive than that of the consumption good (2). By this notation, total labour supply at time t can be written as $L_t = N_t\bar{l}$, where $\bar{l}$ shows the fixed amount of labour inelastically supplied by each young. Factor market equilibrium requires that $k_{1t} + k_{2t} = k_t$ and $l_{1t} + l_{2t} = \bar{l}$, which can be normalized to 1 without loss of generality. The demands for labour and capital in each sector are characterized by the first order conditions for profit maximisation. If labour and capital are perfectly mobile across sectors and if both goods are produced, then $r_t = \alpha k_{1t}^{\alpha-1} l_{1t}^{1-\alpha} = p_t \beta k_{2t}^{\beta-1} l_{2t}^{1-\beta}$, and $w_t = (1-\alpha) k_{1t}^{\alpha} l_{1t}^{-\alpha} = p_t(1-\beta) k_{2t}^{\beta} l_{2t}^{-\beta}$. Solution of the producers' problem gives

$$
\begin{aligned}
l_{1t} &= \frac{\delta}{\delta-\epsilon} - \frac{1}{\delta-\epsilon} k_t p_t^{\frac{1}{\beta-\alpha}}, &(6)\\
l_{2t} &= -\frac{\epsilon}{\delta-\epsilon} + \frac{1}{\delta-\epsilon} k_t p_t^{\frac{1}{\beta-\alpha}}, &(7)\\
k_{1t} &= -\frac{\epsilon}{\delta-\epsilon} k_t + \frac{\delta\epsilon}{\delta-\epsilon} p_t^{\frac{1}{\beta-\alpha}}, &(8)\\
k_{2t} &= \frac{\delta}{\delta-\epsilon} k_t - \frac{\delta\epsilon}{\delta-\epsilon} p_t^{\frac{1}{\beta-\alpha}}, &(9)
\end{aligned}
$$

where

$$
\begin{aligned}
\epsilon &= \left(\frac{\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \left(\frac{1-\beta}{1-\alpha}\right)^{\frac{1-\beta}{\alpha-\beta}}, &(10)\\
\delta &= \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{1-\beta}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha-\beta}}. &(11)
\end{aligned}
$$

Hence,

$$r_t = \alpha\epsilon^{\alpha-1}p_t^{\frac{\alpha-1}{\alpha-\beta}} = \beta\delta^{\beta-1}p_t^{\frac{\alpha-1}{\alpha-\beta}}, \tag{12}$$

$$w_t = (1-\alpha)\epsilon^{\alpha}p_t^{\frac{\alpha}{\alpha-\beta}} = (1-\beta)\delta^{\beta}p_t^{\frac{\alpha}{\alpha-\beta}}, \tag{13}$$

and per capita outputs can now be written as $x_{1t} = l_{1t}\epsilon^{\alpha}p_t^{\frac{\alpha}{\alpha-\beta}}$, and $x_{2t} = l_{2t}\delta^{\beta}p_t^{\frac{\beta}{\alpha-\beta}}$.

2.3 The Autarky Equilibrium

A perfect-foresight equilibrium is a sequence $\{k_t, p_t\}_{t=0}^{\infty}$ that clears the goods' markets at every period t, while satisfying the dynamics of the capital stock at time $t+1$. Remembering that the fraction of income saved during the first period of life is $(1-\mu)$, the evolution of per capita capital is governed by

$$k_{t+1} = \frac{(1-\mu)w_t}{(1+n)}. \tag{14}$$

The clearance of the goods' market in period t requires that per capita supply of each good be equal to its respective per capita demand. Hence,

$$x_{1t} = c_{1yt} + \frac{1}{(1+n)}c_{1ot} + (1+n)k_{t+1} - k_t, \tag{15}$$

$$x_{2t} = c_{2yt} + \frac{1}{(1+n)}c_{2ot}. \tag{16}$$

Walras' law allows us to focus on the market clearance condition for the consumption good (good 2) alone. Substituting c_{2yt} and c_{2ot} from (3) and (5), using (7), (12), (13), and remembering that $x_{2t} = l_{2t}\delta^{\beta}p_t^{\frac{\beta}{\alpha-\beta}}$, one obtains

$$k_t = \phi_1 p_t^{\frac{1}{\alpha-\beta}} + \phi_2 p_{t-1}^{\frac{\alpha}{\alpha-\beta}} p_t^{\frac{1-\alpha}{\alpha-\beta}} + \phi_3 p_{t-1}^{\frac{\alpha}{\alpha-\beta}}, \tag{17}$$

where

$$\phi_1 = \mu(1-\theta)(1-\beta)(\delta-\epsilon) + \epsilon, \tag{18}$$

$$\phi_2 = \frac{1}{1+n}(1-\mu)(1-\theta)(1-\beta)(\delta-\epsilon) \tag{19}$$

$$\phi_3 = \frac{1}{1+n}(1-\mu)(1-\theta)(1-\beta)(\delta-\epsilon)\beta\delta^{\beta-1}. \tag{20}$$

Now, substituting (13) into (14), per capita capital dynamics equation can simply be written as

$$k_{t+1} = \phi_4 p_t^{\frac{\alpha}{\alpha-\beta}}, \tag{21}$$

where

$$\phi_4 = \frac{1}{1+n}(1-\mu)(1-\beta)\delta^{\beta}. \tag{22}$$

Remembering (17) and using (21) one can obtain a nonlinear difference equation in terms of price ratios only. This equation characterizes the dynamics of our model economy and is given by

$$(\phi_4 - \phi_3) p_t^{\frac{\alpha}{\alpha-\beta}} = \phi_1 p_{t+1}^{\frac{1}{\alpha-\beta}} + \phi_2 p_t^{\frac{\alpha}{\alpha-\beta}} p_{t+1}^{\frac{1-\alpha}{\alpha-\beta}}. \tag{23}$$

2.3.1 Steady-State Values of Key Variables under Autarky

The equilibrium steady state value of p_s satisfies (23) with $p_{t+1} = p_t = p_s$. Ruling out $p_s = 0$, p_s is given by

$$p_s = \Phi^{\frac{\alpha-\beta}{1-\alpha}}, \quad \text{where} \quad \Phi = \frac{\phi_4 - \phi_3}{\phi_1 + \phi_2}, \tag{24}$$

as shown in the Appendix.

Proposition 1 *The equilibrium price ratio, p_s, for this perfect foresight overlapping-generations general equilibrium model with constant returns to scale production exists and is unique for all values of $-1 < n$ and given values of α, β, μ, θ that lie strictly between* 0 *and* 1 *such that $\alpha > \beta$.*

Proof:

Straightforward: Uniqueness follows from the closed form solution for p_s in (24), and the existence is assured by the fact that $\Phi > 0$ for the given parameters. ∎

Consequently, the closed form solutions for the steady state per capita values are obtained as

$$
\begin{aligned}
k_s &= \phi_4 \Phi^{\frac{\alpha}{1-\alpha}}, && (25)\\
w_s &= (1-\alpha)\epsilon^{\alpha}\Phi^{\frac{\alpha}{1-\alpha}}, && (26)\\
r_s &= \alpha\epsilon^{\alpha-1}\frac{1}{\Phi}, && (27)\\
c_{1ys} &= \mu\theta(1-\alpha)\epsilon^{\alpha}\Phi^{\frac{\alpha}{1-\alpha}}, && (28)\\
c_{2ys} &= \mu(1-\theta)(1-\alpha)\epsilon^{\alpha}\Phi^{\frac{\beta}{1-\alpha}}, && (29)\\
c_{1os} &= (1-\mu)\theta(1-\alpha)\epsilon^{\alpha}(1+\alpha\epsilon^{\alpha-1}\frac{1}{\Phi})\Phi^{\frac{\alpha}{1-\alpha}}, && (30)\\
c_{2os} &= (1-\mu)(1-\theta)(1-\alpha)\epsilon^{\alpha}(1+\alpha\epsilon^{\alpha-1}\frac{1}{\Phi})\Phi^{\frac{\beta}{1-\alpha}}. && (31)
\end{aligned}
$$

2.3.2 Population Growth Rates and Comparative Advantages

Corollary 1 *The equilibrium relative price ratio, p_s, is decreasing in the population growth rate n.*

The effect of the population growth rate, n, on the steady state price ratio is given by

$$\frac{\partial p_s}{\partial n} = \left(\frac{\alpha-\beta}{1-\alpha}\right)\Phi^{\frac{\alpha-\beta}{1-\alpha}-1}\frac{\partial \Phi}{\partial n}. \qquad (32)$$

Since $\frac{\partial \Phi}{\partial n} < 0$ (see Appendix),

$$\frac{\partial p_s}{\partial n} < 0 \qquad \text{for } \alpha > \beta. \qquad (33)$$

Thus, the equilibrium price of consumption good 2 decreases with n, implying that the autarky value of p_s will be lower in the country with a higher population growth rate.

Given that production of good 2 is relatively labour-intensive, one can conclude that the country with a rapidly growing population will have a relative cost advantage in the production of labour-intensive commodities, whereas the country with a slowly growing population will have a relative cost advantage in the production of capital-intensive commodities. In other words, if we start with two countries that are identical in every respect except the population growth rates, the high-(low-)population growth country will become labour-(capital-)abundant over time, and have a comparative advantage/specialise in the production of labour-(capital) intensive commodity, just as predicted by the static HO model.

Corollary 2 *The steady state values of per capita capital, k_s, and the wage rate, w_s, are decreasing in the population growth rate n, whereas that of the rental rate, r_s, is increasing in the population growth rate n.*

The effect of the population growth rate on the steady state value of per capita capital can be seen from

$$\frac{\partial k_s}{\partial n} = \Phi^{\frac{\alpha}{1-\alpha}} \frac{\partial \phi_4}{\partial n} + \phi_4 \left(\frac{\alpha}{1-\alpha} \right) \Phi^{\frac{\alpha}{1-\alpha}-1} \frac{\partial \Phi}{\partial n}. \tag{34}$$

Since $\phi_4 > 0$, $\Phi > 0$ and $\frac{\partial \phi_4}{\partial n} = -\frac{\phi_4}{1+n} < 0$, and $\frac{\partial \Phi}{\partial n} < 0$, $\frac{\partial k_s}{\partial n} < 0$. Thus, the long-run stock of capital per capita decreases as n increases.

The effect of the population growth rate on the steady state wage rate, w_s, depends on the sign of

$$\frac{\partial w_s}{\partial n} = \alpha \epsilon^{\alpha} \Phi^{\frac{\alpha}{1-\alpha}-1} \frac{\partial \Phi}{\partial n}. \tag{35}$$

Since $\frac{\partial \Phi}{\partial n} < 0$, $\frac{\partial w_s}{\partial n} < 0$. This means that low-population growth countries would have a higher wage rate than high-population growth countries, explaining why they would have a comparative disadvantage in the production of labour-intensive commodities. This also implies that unequal population growth rates could induce labour-migration from high- to low-population growth nations in the absence of barriers to labour mobility (Sayan [8]).

The effect of the population growth rate on the steady state rental rate, r_s, can be observed through

$$\frac{\partial r_s}{\partial n} = \alpha \epsilon^{\alpha-1} \frac{\partial}{\partial n} \left(\frac{1}{\Phi} \right). \tag{36}$$

which is always positive, since $\frac{\partial}{\partial n} \left(\frac{1}{\Phi} \right) > 0$ (see Appendix). Hence, countries with a slowly growing population tend to have a lower rental rate on capital than countries with a rapidly growing population. This is what gives these countries a comparative advantage in the production of capital-intensive commodities, and, in the absence of restrictions to capital mobility, would encourage flows of capital from capital-abundant countries to labour-abundant countries. Furthermore, capital flows induced by population ageing in one region of the world can transmit the growth and resource allocation effects of ageing globally, as suggested before by Tosun [10] and Kenc and Sayan [6].

Corollary 3 *The equilibrium per capita consumptions by youngs of good* 1*,* c_{1ys}*, and good* 2*,* c_{2ys}*, are decreasing in the population growth rate* n*, whereas the equilibrium per capita consumptions by the elderly of both goods are ambiguous in the population growth rate,* n*.*

The first period equilibrium consumptions of both goods decrease in the population growth rate. This inverse relationship between n and equilibrium values of young generation's consumption follows from the negative relationship between the wage rate and n in the case of good 1, and from the fact that the population growth rate elasticity of the price ratio is higher than the population growth rate elasticity of the wage rate in the case of good 2.

The second period equilibrium per capita consumption of good 1 is decreasing in n, if $\Phi > (1-2\alpha)\epsilon^{\alpha-1}$, and is increasing in n, if $\Phi < (1-2\alpha)\epsilon^{\alpha-1}$. Similarly, the second period per capita consumption of good 2 is decreasing in n, if $\Phi > \frac{\alpha}{\beta}(1-\alpha-\beta)\epsilon^{\alpha-1}$, and is increasing in n, if $\Phi < \frac{\alpha}{\beta}(1-\alpha-\beta)\epsilon^{\alpha-1}$.[3]

As previously suggested by Jelassi and Sayan [5] and discussed in the following sections, this uncertainty is, in fact, the reason why welfare effects of trade between two countries would not be as straightforward to predict as in the case of the static HO model.

[3] See Section 5.4 in the Appendix for detailed derivations of the relationship between steady state values of consumption variables and population growth rate, n.

3 Trade

We now suppose that the world is made up of two countries of the type described above. We denote the countries S and F, and assume that they are similar in every respect except for the population growth rates. We let the growth rate of slowly growing population of country S be n^S and that of fast growing population of country F be n^F. Opening of trade sets the worldwide demand for each good equal to the respective worldwide supply. Hence, the world market clearing condition for good 1 is given by

$$\sum_i X_{1t}^i = \sum_i \left(K_{t+1}^i - K_t^i\right) + \sum_i \left(C_{1yt}^i + C_{1ot}^i\right), \qquad \text{for} \quad i = S, F. \tag{37}$$

and the world market clearing condition for good 2 is given by

$$\sum_i X_{2t}^i = \sum_i \left(C_{2yt}^i + C_{2ot}^i\right), \qquad \text{for} \quad i = S, F. \tag{38}$$

where X_{jt}^i is total output of sector j $(j = 1, 2)$ in country i $(i = S, F)$; K_t^i is capital stock in country i; C_{jyt}^i is total consumption of good j by the young in country i, and C_{jot}^i is total consumption of good j by the old in country i, all at time t.

Walras' law allows us to focus on the market clearance condition for the consumption good (good 2) alone. Rewriting (38) in per capita terms results in

$$N_t^S x_{2t}^S + N_t^F x_{2t}^F = N_t^S c_{2yt}^S + N_{t-1}^S c_{2ot}^S + N_t^F c_{2yt}^F + N_{t-1}^F c_{2ot}^F, \tag{39}$$

where N_t^i is population size of the young at time t in country i, and N_{t-1}^i is population size of the old at time t in country i. Given that free trade will lead to an equalization of prices in both countries in each period, $p_t^* = p_t^S = p_t^F$, where p_t^i is the price of good 2 in terms of good 1, must hold true for every t. Similarly to the case of autarky, the dynamics equation characterizing the world economy is now determined to be

$$\left(\sum_i (1+n^i)^t\right)(\bar{\phi}_4 - \bar{\phi}_3) p_t^{\frac{\alpha}{\alpha-\beta}} = \left(\sum_i (1+n^i)^{t+1}\right)\phi_1 p_{t+1}^{\frac{1}{\alpha-\beta}} + \left(\sum_i (1+n^i)^t\right)\bar{\phi}_2 p_t^{\frac{\alpha}{\alpha-\beta}} p_{t+1}^{\frac{1-\alpha}{\alpha-\beta}}, \tag{40}$$

where

$$\begin{aligned}
\phi_1 &= \mu(1-\theta)(1-\beta)(\delta-\epsilon)+\epsilon, && (41)\\
\bar{\phi}_2 &= (1-\mu)(1-\theta)(1-\beta)(\delta-\epsilon), && (42)\\
\bar{\phi}_3 &= (1-\mu)(1-\theta)(1-\beta)(\delta-\epsilon)\beta\delta^{\beta-1}, && (43)\\
\bar{\phi}_4 &= (1-\mu)(1-\beta)\delta^{\beta}. && (44)
\end{aligned}$$

3.1 Closed Form Solutions under Trade

The equilibrium steady state world price ratio value p_s^* satisfies (40) with $p_{t+1} = p_t = p_s^*$. Ruling out $p_s^* = 0$, p_s^* is given by

$$p_s^* = \bar{\Phi}^{\frac{\alpha-\beta}{1-\alpha}}, \quad \text{where} \quad \bar{\Phi} = \frac{(\sum_i (1+n^i)^s)(\bar{\phi}_4 - \bar{\phi}_3)}{(\sum_i (1+n^i)^{s+1})\phi_1 + (\sum_i (1+n^i)^s)\bar{\phi}_2}. \tag{45}$$

Proposition 2 *The equilibrium price ratio, p_s^*, for this perfect foresight world economy model exists and is unique for all $n^F > n^S > -1$ and given values of α, β, μ, θ that lie strictly between 0 and 1 such that $\alpha > \beta$, and satisfy the interior solution condition.*

Proof:

Straightforward: Uniqueness follows from the closed form solution for p_s^* in (45), and the existence is assured by the fact that $\bar{\Phi} > 0$ for the given parameters. An interior solution with $x_{1s,i}^* > 0$ and $x_{2s,i}^* > 0$ for $i = S, F$ would further require that $\frac{(1-\mu)(1-\alpha)}{(1+n^i)}\epsilon^{\alpha-1} < \bar{\Phi} < \frac{\alpha}{\beta}\frac{(1-\mu)(1-\beta)}{(1+n^i)}\epsilon^{\alpha-1}$. ∎

Thus, closed form solutions for the steady state per capita values will be

$$\begin{aligned}
k_{i,s}^* &= \frac{1}{1+n^i}\bar{\phi}_4\bar{\Phi}^{\frac{\alpha}{1-\alpha}} \qquad \text{for} \qquad i = S, F, && (46)\\
w_s^* &= (1-\alpha)\epsilon^{\alpha}\bar{\Phi}^{\frac{\alpha}{1-\alpha}}, && (47)\\
r_s^* &= \alpha\epsilon^{\alpha-1}\frac{1}{\bar{\Phi}}, && (48)\\
c_{1ys}^* &= \mu\theta(1-\alpha)\epsilon^{\alpha}\bar{\Phi}^{\frac{\alpha}{1-\alpha}}, && (49)\\
c_{2ys}^* &= \mu(1-\theta)(1-\alpha)\epsilon^{\alpha}\bar{\Phi}^{\frac{\beta}{1-\alpha}}, && (50)
\end{aligned}$$

$$c^*_{1os} \quad = \quad (1-\mu)\theta(1-\alpha)\epsilon^\alpha(1+\alpha\epsilon^{\alpha-1}\frac{1}{\bar{\bar{\Phi}}})\bar{\Phi}^{\frac{\alpha}{1-\alpha}}, \tag{51}$$

$$c^*_{2os} \quad = \quad (1-\mu)(1-\theta)(1-\alpha)\epsilon^\alpha(1+\alpha\epsilon^{\alpha-1}\frac{1}{\bar{\bar{\Phi}}})\bar{\Phi}^{\frac{\beta}{1-\alpha}}. \tag{52}$$

3.2 The Role of Population Growth Rate under Trade

Corollary 4 *Free trade increases the autarky relative price of labour-intensive commodity 2 in the slow-population growth (capital-abundant) country S, and lowers it in the fast-population growth (labour-abundant) country F.*

This can be easily seen by rewriting the expressions for the steady state relative prices under autarky as

$$p^i_s = (\bar{\phi}_4 - \bar{\phi}_3)^{\frac{\alpha-\beta}{1-\alpha}} \left(\bar{\phi}_2 + \phi_1(1+n^i)\right)^{\frac{\beta-\alpha}{1-\alpha}}, \qquad \text{for} \qquad i = S, F, \tag{53}$$

and the common relative price under trade as

$$p^*_s = (\bar{\phi}_4 - \bar{\phi}_3)^{\frac{\alpha-\beta}{1-\alpha}} \left(\bar{\phi}_2 + \phi_1 \frac{(1+n^F)^{s+1} + (1+n^S)^{s+1}}{(1+n^F)^s + (1+n^S)^s}\right)^{\frac{\beta-\alpha}{1-\alpha}}. \tag{54}$$

Since $n^F > n^S$ and $\alpha > \beta$, $p^F_s < p^*_s < p^S_s$. This equation also indicates that the higher the difference between the population growth rates is, the lesser the increase in the relative price of commodity 2 for the high-population growth rate country will be. This is a significant finding, since it implies that for a high enough difference between n^F and n^S, country F may start acting as a large country that is capable of setting the terms of trade close to its autarky relative price ratio.

Corollary 5 *Free trade leads in the long-run to*

- *an increase in (the steady state values of) per capita capital stock and the wage rate, and a decrease in (the steady state value of) the rental rate in the high-population growth rate country, and*

- *a decrease in the per capita capital, a decrease in the wage rate and an increase in the rental rate for the low-population growth rate country.*

The steady state expressions for per capita capital can be rewritten as

$$k_s^i = \frac{\bar{\phi}_4}{1+n^i}(\bar{\phi}_4 - \bar{\phi}_3)^{\frac{\alpha}{1-\alpha}} \left(\bar{\phi}_2 + \phi_1(1+n^i)\right)^{-\frac{\alpha}{1-\alpha}} \qquad \text{for} \qquad i = S, F, \tag{55}$$

under autarky, and as

$$k_{i,s}^* = \frac{\bar{\phi}_4}{1+n^i}(\bar{\phi}_4 - \bar{\phi}_3)^{\frac{\alpha}{1-\alpha}} \left(\bar{\phi}_2 + \phi_1 \frac{(1+n^F)^{s+1} + (1+n^S)^{s+1}}{(1+n^F)^s + (1+n^S)^s}\right)^{-\frac{\alpha}{1-\alpha}} \qquad \text{for} \qquad i = S, F, \tag{56}$$

under trade. Since $n^F > n^S$, $k_s^F < k_s^S$ and $k_{F,s}^* < k_{S,s}^*$. It can also be shown that $k_s^F < k_{F,s}^*$ and $k_{S,s}^* < k_s^S$. This equation further implies that the higher the difference between population growth rates is, the smaller (larger) the effect on the per capita capital stock of the high-population growth country, F (low-population growth country, S) will be.

The steady state wage rate under autarky is given by

$$w_s^i = (1-\alpha)\epsilon^\alpha(\bar{\phi}_4 - \bar{\phi}_3)^{\frac{\alpha}{1-\alpha}} \left(\bar{\phi}_2 + \phi_1(1+n^i)\right)^{-\frac{\alpha}{1-\alpha}} \qquad \text{for} \qquad i = S, F. \tag{57}$$

Since $n^F > n^S$, $w_s^F < w_s^S$. The wage rate under trade, on the other hand, is given by

$$w_s^* = (1-\alpha)\epsilon^\alpha(\bar{\phi}_4 - \bar{\phi}_3)^{\frac{\alpha}{1-\alpha}} \left(\bar{\phi}_2 + \phi_1 \frac{(1+n^F)^{s+1} + (1+n^S)^{s+1}}{(1+n^F)^s + (1+n^S)^s}\right)^{-\frac{\alpha}{1-\alpha}}, \tag{58}$$

clearly indicating that $w_s^F < w_s^* < w_s^S$. In addition, equation (58) implies that the reduction in the autarky wage rate experienced by the slow-population growth country S will be relatively higher than the increase that the fast-population growth country F would observe in its autarky wage rate after opening of trade. In other words, trade would lead to a proportionately smaller change in the autarky value of the wage rate in country F than in country S.

As for the rental rate, the steady state value would be given by

$$r_s^i = \alpha\epsilon^{\alpha-1}(\bar{\phi}_4 - \bar{\phi}_3)^{-1} \left(\bar{\phi}_2 + \phi_1(1+n^i)\right) \qquad \text{for} \qquad i = S, F, \tag{59}$$

under autarky, and by

$$r_s^* = \alpha\epsilon^{\alpha-1}(\bar{\phi}_4 - \bar{\phi}_3)^{-1} \left(\bar{\phi}_2 + \phi_1 \frac{(1+n^F)^{s+1} + (1+n^S)^{s+1}}{(1+n^F)^s + (1+n^S)^s}\right), \tag{60}$$

under trade. Since $n^F > n^S$, $r_s^F > r_s^S$. Once again, the common rental rate after trade would settle between autarky values such that $r_s^F > r_s^* > r_s^S$. It is now straightforward to see that the higher the difference in the population growth rates between the trading partners is, the higher the effect of trade on the rental rate of the low-population growth rate country will be.

Thus, our model suggests, in line with the expectations based on the solution of the autarky model as discussed in the previous section, that the high-population growth rate country (F) will have a comparative advantage in the production of labour-intensive consumption good 2, and the low-population growth rate country (S) will have a comparative advantage in the production of capital-intensive good 1 that serves as both an investment good and a consumption good. Furthermore, relative commodity and factor prices under trade will lie between corresponding autarky values just as in the static Heckscher-Ohlin framework, but be closer to the pre-trade values for country F in magnitude. This implies that trade creates a tendency for the high-population growth rate country F to pull the values of all variables towards its own steady state autarky values in the long-run. In fact, the larger the difference between population growth rates is, the stronger this tendency will get, enabling country F to behave as the large country setting the terms of trade in the long-run.

The challenges that remain now are i) to show that trade may continue to occur in the long-run despite the equalization of these prices under trade, as long as population growth rates are different, and ii) to compare welfare levels across autarky and trade scenarios.

Corollary 6 *The nations considered may continue trading in the long-run as a result of the differences in population growth rates alone, as the initial pattern of comparative advantages are preserved at the steady state.*

To prove Corollary 6, it suffices to show that domestic markets will not clear. Instead, each country will have an excess supply of one commodity (to be exported), and an excess demand for the other (to be satisfied through imports). Here, we show only the long-run expression for the excess supply of good 2 by country F. This is given by

$$Exs_2^H = (1-\alpha)\epsilon^\alpha \bar{\Phi}^{\frac{\beta}{1-\alpha}} N_s^F \left\{ -\frac{\alpha}{\beta-\alpha} - (1-\theta)\left(\mu + \frac{1-\mu}{1+n^F}\right) \right. \\ \left. + \quad \frac{1-\mu}{1+n^F}\alpha\epsilon^{\alpha-1}\bar{\Phi}^{-1}\left(\frac{1-\alpha}{\beta-\alpha}\right) \right\} \tag{61}$$

which is clearly positive for $\alpha > \beta$. This implies that the country with a fast growing population (i.e., the labour-abundant-country) will export the labour-intensive commodity 2, as expected.

Having shown that trade may continue to occur at the steady state, we can now compare welfare levels across autarky and trade. Again, as expected from solutions under autarky, it is not obvious that welfare results are consistent with the static HO model.

Corollary 7 *Free trade leads to*

- *a decrease in the per capita consumption by youngs of good* 1 *and good* 2 *in the low-population growth rate country* S*, and an increase in the per capita consumption by youngs of both goods in the high-population growth rate country,* F*.*

- *an ambiguous effect on the per capita consumption by olds of both goods in both countries.*

The long-run equilibrium value of per capita consumption of good 1 by the youngs is given by $c_{1ys}^i = \mu\theta w_s^i$ for $i = S, F$ under autarky, and by $c_{1ys}^* = \mu\theta w_s^*$ under trade. Since $w_s^F < w_s^* < w_s^S$, $c_{1ys}^F < c_{1ys}^* < c_{1ys}^S$ and a similar ranking can be made for good 2. The long-run equilibrium value of per capita consumption by the youngs is given by $c_{2ys}^i = \mu(1-\theta)\frac{w_s^i}{p_s^i}$ for $i = S, F$ under autarky, and by $c_{2ys}^* = \mu(1-\theta)\frac{w_s^*}{p_s^*}$ under trade. Since the real wage rate has been shown to decrease in the population growth rate, $\frac{w_s^F}{p_s^F} < \frac{w_s^*}{p_s^*} < \frac{w_s^S}{p_s^S}$ implying that $c_{2ys}^F < c_{2ys}^* < c_{2ys}^S$.

Thus, trade leads to an increase (decrease) in the youngs' consumption of both commodities in country F (S) in per capita terms.

Such a ranking, however, is not easy to find in the case of per capita consumption of goods by the olds. For commodity 1, per capita consumption by the olds under autarky is given by $c_{1os} = (1-\mu)\theta(1+r_s^i)w_s^i$ for $i = S, F$. Remembering that trade leads to a decrease (an increase) in the long-run wage rate but an increase (a dcrease) in the long-run rental rate for the slow-population growth (fast-population growth) country, the overall effect of trade on c_{1os}^i, depends on whether the magnitude of the population growth rate elasticity of gross rental rate is smaller or greater than the magnitude of the population growth rate elasticity of the wage rate.

Similarly, since per capita consumption of good 2 by the olds under autarky is given by $c_{2os}^i = (1-\mu)(1-\theta)(1+r_s^i)\frac{w_s^i}{p_s^i}$ for $i = S, F$, the overall effect of trade on c_{2os}^i would also depend the relative magnitudes of the population growth rate elasticities of gross rental rate and the real wage rate. Thus, the welfare of the low-population growth rate country may increase or decrease depending upon the sign of the following derivative (see Appendix):

$$\begin{aligned} e_{u_s,n} &= \mu\theta e_{c_{1ys},n} + \mu(1-\theta)e_{c_{2ys},n} + (1-\mu)\theta e_{c_{1os},n} + (1-\mu)(1-\theta)e_{c_{2os},n}, \\ &= \theta e_{w_s,n} + (1-\theta)e_{\frac{w_s}{p_s},n} + (1-\mu)e_{(1+r_s),n} \qquad (62) \end{aligned}$$

It is conceivable that $e_{u_s,n} < 0$ may hold, unless there are additional restrictions on the values of parameters. Therefore, a country with a low population growth rate may very well face a reduction in its autarky level of welfare after beginning to trade with a high-population growth rate country.

4 Conclusions

Our discussion of the closed form solutions to the 2x2 and 2x2x2 OLG models in the paper has shown that of the two countries/regions that are identical in every respect except the population growth rates, the high-(low-) population growth country will become labour-(capital-) abundant over time, and must be expected to have a comparative advantage in the production of labour-(capital-) intensive commodity, as suggested by the static HO model. Furthermore, we have shown that as long as the population growth rates are different, there will be room for trade to continue to occur in the long-run. Unlike what static HO model predicts, however, trade will not necessarily make both countries better off in the long-run and our analysis has revealed the reasons underlying this.

The analysis in Section 2 has shown that when the model is solved under autarky, differences in the population growth rates alone are observed to give rise to comparative advantages by leading to different relative prices across countries, regardless of initial population sizes of trading countries. In other words, the only difference in demographic characteristics that matters for the direction of product and factor flows is the one between population growth rates. An examination of the sensitivity of the steady state value of relative price ratio under autarky to changes in population growth rate will indeed identify directions of comparative advantages correctly.

While the population size does not directly affect the long-run equilibrium, the discussion in Section 3 about the changes that the opening of trade introduced to relative commodity and factor prices prevailing under autarky hinted that initial population sizes of trading nations may play a role in determining the gains from trade. That's because trade creates a tendency for the high-population growth rate country F to pull the values of all variables towards its own steady state autarky values in the long-run. In fact, we have shown that this tendency will get stronger, the larger the difference between population growth rates. This implies that country F has the potential to behave as a large country capable of setting the terms of trade in the long-run, as a result of the parallel growth in its share of total world output and population. Symmetrically, the low-population growth rate country S will become a small country, and will begin to act as a price taker in trade. Thus, unless there is a large enough differential in initial population sizes, fast growing population of country F will soon overtake country S in size, and the diverging population sizes will lead to a divergence in the shares of countries in total world output. If the resulting difference between these shares becomes sufficiently large before the steady state is reached, country F will begin to act as the price setter, thereby driving all results. Such a dominance will cause welfare of the world to converge to the autarky welfare of country F, creating welfare losses for country S. It would therefore be correct to argue that initial population size would matter in determining the nature of gains from trade, even though direction of trade itself is determined by differences in population growth rates alone

In summary, the analysis in this paper has shown that demographically induced differences in relative endowments by themselves may not be sufficient for trade to be beneficial to both parties in the long-run, and offered a new explanation for this, adding to previously suggested reasons as to why trade may not be Pareto-superior to autarky in a dynamic, OLG set-up.

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Appendix

5.1 The steady state price ratio

Proof:

The analytical solution of the steady state price ratio p_s can easily be obtained by rearranging terms of (23). This results in

$$\phi_4 - \phi_3 = \left(\phi_1 \left(\frac{p_{t+1}}{p_t}\right)^{\frac{\alpha}{\alpha-\beta}} + \phi_2\right) p_{t+1}^{\frac{1-\alpha}{\alpha-\beta}}. \tag{63}$$

Hence,

$$p_{t+1} = \left(\frac{\phi_4 - \phi_3}{\phi_1 \left(\frac{p_{t+1}}{p_t}\right)^{\frac{\alpha}{\alpha-\beta}} + \phi_2}\right)^{\frac{\alpha-\beta}{1-\alpha}}. \tag{64}$$

Since an equilibrium value p_s (steady state value) is such that $p_{t+1} = p_t = p_s$ and satisfies (23) it follows that

$$p_s = \Phi^{\frac{\alpha-\beta}{1-\alpha}}, \quad \text{where} \quad \Phi = \tfrac{\phi_4-\phi_3}{\phi_1+\phi_2}. \tag{65}$$

Now, (65) shows that p_s is unique for any given set of parameter values. For existence, we also need to show that p_s is positive for any $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \mu < 1$, $0 < \theta < 1$ and $-1 < n$. $\Phi > 0$ if and only if $\phi_4 - \phi_3 > 0$ and $\phi_1 + \phi_2 > 0$ or $\phi_4 - \phi_3 < 0$ and $\phi_1 + \phi_2 < 0$. Now, $\phi_4 - \phi_3 > 0 \Leftrightarrow$

$$\begin{aligned}
\frac{1}{1+n}(1-\mu)(1-\beta)\delta^{\beta} &> \frac{1}{1+n}(1-\mu)(1-\theta)(1-\beta)\beta(\delta-\epsilon)\delta^{\beta-1} \\
\delta^{\beta} &> (1-\theta)\beta(\delta-\epsilon)\delta^{\beta-1} \\
1 &> (1-\theta)\beta(1-\frac{\epsilon}{\delta}) \\
1 &> (1-\theta)\beta\left(1-\left(\frac{1-\beta}{1-\alpha}\right)\left(\frac{\alpha}{\beta}\right)\right) \\
1 &> \frac{(1-\theta)(\beta-\alpha)}{(1-\alpha)} \\
(1-\alpha)+\alpha(1-\theta) &> (1-\theta)\beta \\
\frac{1-\alpha\theta}{1-\theta} &> \beta
\end{aligned}$$

since $\frac{1-\alpha\theta}{1-\theta} > 1$, and $0 < \beta < 1$. Then, $\frac{1-\alpha\theta}{1-\theta} > \beta$ holds for any given α, β and θ. Thus, $\phi_4 - \phi_3 > 0$ for any given $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \mu < 1$, $0 < \theta < 1$ and $-1 < n$.

Similarly, $\phi_1 + \phi_2 > 0 \Leftrightarrow$

$$\begin{aligned}
\mu(1-\theta)(1-\beta)(\delta-\epsilon) + \epsilon + \frac{1}{1+n}(1-\mu)(1-\theta)(1-\beta)(\delta-\epsilon) &> 0 \\
(\delta-\epsilon)\left(\mu(1-\theta)(1-\beta) + \frac{1}{1+n}(1-\mu)(1-\theta)(1-\beta)\right) + \epsilon &> 0 \\
\left(\frac{\delta}{\epsilon} - 1\right)(1-\theta)(1-\beta)\left(\mu + \frac{1-\mu}{1+n}\right) &> -1 \\
\left(\frac{\beta-\alpha}{\alpha}\right)(1-\theta)\left(\frac{1+\mu n}{1+n}\right) &> -1 \\
-\frac{1+n}{(1+\mu n)(1-\theta)} &< \frac{\beta-\alpha}{\alpha} \\
1 - \frac{1+n}{(1+\mu n)(1-\theta)} &< \frac{\beta}{\alpha}
\end{aligned}$$

Since $\frac{1+n}{(1+\mu n)(1-\theta)} > 1$, $1 - \frac{1+n}{(1+\mu n)(1-\theta)} < 0$. But $\frac{\beta}{\alpha} > 0$, and hence, $\frac{\beta}{\alpha} > 1 - \frac{1+n}{(1+\mu n)(1-\theta)}$ holds for any given values of α, β, μ, θ and n. Thus, $\phi_1 + \phi_2 > 0$ for any given α, β, μ, θ, and n. Therefore, $p_s > 0$ for any given α, β, μ, θ, $\bar{l}$ and n, where $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \mu < 1$, $0 < \theta < 1$, and $-1 < n$. ∎

5.2 the sign of $\frac{\partial \Phi}{\partial n}$

First of all,

$$\begin{aligned}
\phi_4 - \phi_3 &= \frac{1}{1+n}(1-\mu)(1-\beta)\left(\delta^\beta - (1-\theta)\beta(\delta-\epsilon)\delta^{\beta-1}\right), \\
\phi_1 + \phi_2 &= (1-\theta)(1-\beta)(\delta-\epsilon)\left(\mu + \frac{1-\mu}{1+n}\right) + \epsilon.
\end{aligned}$$

So,

$$\begin{aligned}
\Phi = \frac{\phi_4 - \phi_3}{\phi_1 + \phi_2} &= \frac{(1-\mu)(1-\beta)(\delta^\beta - (1-\theta)\beta(\delta-\epsilon)\delta^{\beta-1})}{(1+n)((1-\theta)(1-\beta)(\delta-\epsilon)(\mu + \frac{1-\mu}{1+n}) + \epsilon)} \\
&= \frac{(1-\mu)(1-\beta)\delta^\beta[1 - (1-\theta)\beta(1-\frac{\epsilon}{\delta})]}{(1-\theta)(1-\beta)(\delta-\epsilon)(1+n\mu) + (1+n)\epsilon}.
\end{aligned}$$

Taking the derivative of the obove with respect to n results in

$$\begin{aligned}\frac{\partial\Phi}{\partial n} &= \frac{-(1-\mu)(1-\beta)\delta^{\beta}[1-(1-\theta)\beta(1-\frac{\epsilon}{\delta})][(1-\theta)(1-\beta)(\delta-\epsilon)\mu+\epsilon]}{[(1-\theta)(1-\beta)(\delta-\epsilon)(1+n\mu)+(1+n)\epsilon]^2}\\ &= \frac{-(1-\mu)(1-\beta)\frac{\delta^{\beta}}{\epsilon}[1-(1-\theta)\beta(1-\frac{\epsilon}{\delta})][(1-\theta)(1-\beta)(\frac{\delta}{\epsilon}-1)\mu+1]}{[(1-\theta)(1-\beta)(\frac{\delta}{\epsilon}-1)(1+n\mu)+(1+n)]^2}.\end{aligned}$$

Now,

$$1-(1-\theta)\beta(1-\frac{\epsilon}{\delta}) = 1-\frac{(1-\theta)(\beta-\alpha)}{(1-\alpha)}.$$

We have $\frac{\beta-\alpha}{1-\alpha}<1$, and $0<1-\theta<1$, so, $\frac{(1-\theta)(\beta-\alpha)}{(1-\alpha)}<(1-\theta)<1$. Thus,

$$0<1-\frac{(1-\theta)(\beta-\alpha)}{1-\alpha}.$$

Similarly,

$$(1-\theta)(1-\beta)(\frac{\delta}{\epsilon}-1)\mu+1 = (1-\theta)(\beta-\alpha)\frac{\mu}{\alpha}+1. \tag{66}$$

Since $0<\beta<1$, and $0<\alpha<1$, then $-\alpha<\beta-\alpha<1-\alpha$. We also have $0<(1-\theta)\frac{\mu}{\alpha}$. So, $-\alpha(1-\theta)\frac{\mu}{\alpha}<(\beta-\alpha)(1-\theta)\frac{\mu}{\alpha}$. Thus, $-(1-\theta)\mu<(1-\theta)(\beta-\alpha)\frac{\mu}{\alpha}$. But $(1-\theta)\mu<1$. Hence,

$$0<(1-\theta)(\beta-\alpha)\frac{\mu}{\alpha}+1.$$

Therefore,

$$\frac{\partial\Phi}{\partial n}<0.$$

5.3 The sign of $\frac{\partial}{\partial n}\frac{1}{\Phi}$

$$\frac{\partial}{\partial n}\left(\frac{1}{\Phi}\right) = -\frac{1}{\Phi^2}\frac{\partial\Phi}{\partial n}. \tag{67}$$

Since $\frac{\partial\Phi}{\partial n}<0$, $\frac{\partial}{\partial n}\left(\frac{1}{\Phi}\right)>0$.

5.4 The effect of n on Consumption

5.4.1 Consumption of good 1 by young

We have

$$c1ys = \mu\theta w_t.$$

Taking the derivative with respect to n gives,

$$\frac{\partial c_{1ys}}{\partial n} = \mu\theta\frac{\partial w_s}{\partial n}.$$

Since $\frac{\partial w_s}{\partial n} < 0$, $\frac{\partial c_{1ys}}{\partial n} < 0$.

5.4.2 Consumption of good 2 by young

The first period equilibrium per capita consumption of good 2 is

$$c_{2ys} = \mu(1-\theta)\frac{w_s}{p_s}.$$

Plugging the expressions for w_s and p_s in the above and taking the derivative with respect to the population growth rate n results in

$$\frac{\partial c_{2ys}}{\partial n} = \mu(1-\theta)(1-\alpha)\epsilon^{\alpha}\left(\frac{\beta}{1-\alpha}\right)\Phi^{\frac{beta}{1-\alpha}-1}\frac{\partial\Phi}{\partial n}.$$

Since $\frac{\partial\Phi}{\partial n} < 0$, $\frac{\partial c_{2ys}}{\partial n} < 0$. Thus,

$$\frac{\partial c_{2ys}}{\partial n} = \mu(1-\theta)\frac{1}{p_s^2}\left(p_s\frac{\partial w_s}{\partial n} - w_s\frac{\partial p_s}{\partial n}\right) < 0,$$

implying that

$$\frac{n}{w_s}\frac{\partial w_s}{\partial n} < \frac{n}{p_s}\frac{\partial p_s}{\partial n}.$$

Thus, the population growth rate elasticity of the wage rate is less than the population growth rate elasticity of the price ratio.

5.4.3 Consumption of good 1 by old

$$c_{1os} = (1-\mu)\theta(1-\alpha)\epsilon^{\alpha}(\Phi^{\frac{\alpha}{1-\alpha}} + \alpha\epsilon^{\alpha-1}\Phi^{\frac{\alpha}{1-\alpha}-1}).$$

The sign of $\frac{\partial c_{1os}}{\partial n}$ depends on the sign of $\frac{\partial}{\partial n}(\Phi^{\frac{\alpha}{1-\alpha}} + \alpha\epsilon^{\alpha-1}\Phi^{\frac{\alpha}{1-\alpha}-1})$. So,

$$\begin{aligned}\frac{\partial}{\partial n}(\Phi^{\frac{\alpha}{1-\alpha}} + \alpha\epsilon^{\alpha-1}\Phi^{\frac{\alpha}{1-\alpha}-1}) &= \left(\frac{\alpha}{1-\alpha}\right)\Phi^{\frac{\alpha}{1-\alpha}-1}\frac{\partial\Phi}{\partial n} + \alpha\epsilon^{\alpha-1}\left(\frac{\alpha}{1-\alpha}-1\right)\Phi^{\frac{\alpha}{1-\alpha}-2}\frac{\partial\Phi}{\partial n} \\ &= \left(\frac{\alpha}{1-\alpha}\right)\Phi^{\frac{\alpha}{1-\alpha}-1}\frac{\partial\Phi}{\partial n}[1+\epsilon^{\alpha-1}(2\alpha-1)\Phi^{-1}].\end{aligned}$$

Since $\frac{\partial\Phi}{\partial n} < 0$,

$$\frac{\partial c_{1os}}{\partial n}\begin{cases} < 0 & \text{if } \Phi > (1-2\alpha)\epsilon^{\alpha-1} \\ > 0 & \text{if } \Phi < (1-2\alpha)\epsilon^{\alpha-1} \end{cases}. \quad (68)$$

5.4.4 Consumption of good 2 by old

Similarly,

$$c_{2os} = (1-\mu)(1-\theta)(1-\alpha)\epsilon^{\alpha}(\Phi^{\frac{\beta}{1-\alpha}} + \alpha\epsilon^{\alpha-1}\Phi^{\frac{\beta}{1-\alpha}-1}).$$

The sign of $\frac{\partial c_{2os}}{\partial n}$ depends on the sign of $\frac{\partial}{\partial n}(\Phi^{\frac{\beta}{1-\alpha}} + \alpha\epsilon^{\alpha-1}\Phi^{\frac{\beta}{1-\alpha}-1})$. So,

$$\begin{aligned}\frac{\partial}{\partial n}(\Phi^{\frac{\beta}{1-\alpha}} + \alpha\epsilon^{\alpha-1}\Phi^{\frac{\beta}{1-\alpha}-1}) &= \left(\frac{\beta}{1-\alpha}\right)\Phi^{\frac{\beta}{1-\alpha}-1}\frac{\partial\Phi}{\partial n} + \alpha\epsilon^{\alpha-1}\left(\frac{\beta}{1-\alpha}-1\right)\Phi^{\frac{\beta}{1-\alpha}-2}\frac{\partial\Phi}{\partial n} \\ &= \left(\frac{1}{1-\alpha}\right)\Phi^{\frac{\beta}{1-\alpha}-1}\frac{\partial\Phi}{\partial n}[\beta+\alpha\epsilon^{\alpha-1}(\alpha+\beta-1)\Phi^{-1}].\end{aligned}$$

Since $\frac{\partial\Phi}{\partial n} < 0$,

$$\frac{\partial c_{2os}}{\partial n}\begin{cases} < 0 & \text{if } \Phi > \frac{\alpha}{\beta}(1-\alpha-\beta)\epsilon^{\alpha-1} \\ > 0 & \text{if } \Phi < \frac{\alpha}{\beta}(1-\alpha-\beta)\epsilon^{\alpha-1} \end{cases}. \quad (69)$$

5.5 Trade

5.5.1 Trade vs Autarky

First, we have $n^F > n^S$.

$$\begin{aligned}(1-n^S) - \frac{(1+n^F)^{s+1} + (1+n^S)^{s+1}}{(1+n^F)^s + (1+n^S)^s} &= \frac{(1+n^F)^s}{(1+n^F)^s + (1+n^S)^s}(n^S - n^F) \\ &< 0.\end{aligned}$$

Thus,

$$(1-n^S) < \frac{(1+n^F)^{s+1}+(1+n^S)^{s+1}}{(1+n^F)^s+(1+n^S)^s}, \tag{70}$$

and

$$\begin{aligned}\frac{(1+n^F)^{s+1}+(1+n^S)^{s+1}}{(1+n^F)^s+(1+n^S)^s}-(1-n^F) &= \frac{(1+n^S)^s}{(1+n^F)^s+(1+n^S)^s}(n^S-n^F)\\ &< 0.\end{aligned}$$

Thus,

$$\frac{(1+n^F)^{s+1}+(1+n^S)^{s+1}}{(1+n^F)^s+(1+n^S)^s} < (1-n^F). \tag{71}$$

5.6 Condition for $e_{u_s,n}$ to be positive

First of all, $e_{u_s,n}$ is given by

$$e_{u_s,n} = \left(\frac{\theta\alpha+(1-\theta)\beta}{1-\alpha}-(1-\mu)\frac{\alpha\epsilon^{\alpha-1}\Phi^{-1}}{1+\alpha\epsilon^{\alpha-1}\Phi^{-1}}\right)\frac{n}{\Phi}\frac{\partial\Phi}{\partial n}.$$

Since $\frac{\partial\Phi}{\partial n}<0$, the following condition must hold for $e_{u_s,n}$ to be positive

$$\Phi < \left(\frac{(1-\alpha)(1-\mu)}{\theta\alpha+(1-\theta)\beta}-1\right)\alpha\epsilon^{\alpha-1}. \tag{72}$$

Rewriting the term in parentheses results in

$$(1-\mu) < \frac{1}{1-\alpha}\left(\theta(\alpha-\beta)+\beta\right). \tag{73}$$

Since $0<\alpha<1$, $0<\beta<1$ and $0<\theta<1$, we have $0<\theta(\alpha-\beta)<\theta$ for $\alpha>\beta$. So, $\beta<\theta(\alpha-\beta)+\beta<\theta+\beta$. Since $\frac{1}{1-\alpha}>0$, $\frac{\beta}{1-\alpha}<\frac{1}{1-\alpha}(\theta(\alpha-\beta)+\beta)<\frac{\theta+\beta}{1-\alpha}$.

5.7 Excess supply of good 2 by country F

We have the steady state output of good 2 by country F given by $x^*_{2s,F}=l_{2s,F}\delta^\beta(p^*_s)^{\frac{\beta}{\alpha-\beta}}$. Now, substituting (7), (45) and (46) and using the following

$$\frac{\delta}{\delta-\epsilon}=\frac{\beta(1-\alpha)}{\beta-\alpha}, \qquad \frac{\epsilon}{\delta-\epsilon}=\frac{\alpha(1-\beta)}{\beta-\alpha}, \tag{74}$$

$$\alpha\epsilon^{\alpha-1}=\beta\delta^{\beta-1}, \qquad (1-\alpha)\epsilon^\alpha=(1-\beta)\delta^\beta, \tag{75}$$

we get

$$
\begin{aligned}
x^*_{2s,F} &= -\left(\frac{\epsilon}{\delta-\epsilon}\right)\delta^{\beta}\bar{\Phi}^{\frac{\beta}{1-\alpha}} + \left(\frac{\delta}{\delta-\epsilon}\right)\frac{(1-\mu)(1-\beta)}{(1+n^F)}\delta^{2\beta-1}\bar{\Phi}^{\frac{\alpha+\beta-1}{1-\alpha}} \\
&= -\left(\frac{\alpha(1-\beta)}{\beta-\alpha}\right)\delta^{\beta}\bar{\Phi}^{\frac{\beta}{1-\alpha}} + \left(\frac{\beta(1-\alpha)}{\beta-\alpha}\right)\frac{(1-\mu)(1-\beta)}{(1+n^F)}\delta^{2\beta-1}\bar{\Phi}^{\frac{\alpha+\beta-1}{1-\alpha}} \\
&= -\frac{\alpha(1-\alpha)}{\beta-\alpha}\epsilon^{\alpha}\bar{\Phi}^{\frac{\beta}{1-\alpha}} + \frac{\alpha(1-\alpha)}{\beta-\alpha}\frac{(1-\mu)(1-\beta)}{(1+n^F)}\epsilon^{2\beta-1}\bar{\Phi}^{\frac{\alpha+\beta-1}{1-\alpha}} \\
&= \frac{\alpha}{\beta-\alpha}(1-\alpha)\epsilon^{\alpha}\bar{\Phi}^{\frac{\beta}{1-\alpha}}\left(-1+\frac{(1-\mu)(1-\alpha)}{1+n^F}\epsilon^{\alpha-1}\bar{\Phi}^{-1}\right). \qquad (76)
\end{aligned}
$$

The long-run aggregate consumption is given by

$$
\begin{aligned}
N^F_s c^*_{2ys,F} + N^F_{s-1}c^*_{2os,F} &= N^F_s\mu(1-\theta)(1-\alpha)\epsilon^{\alpha}\bar{\Phi}^{\frac{\beta}{1-\alpha}} \\
&+ N^F_{s-1}(1-\mu)(1-\theta)(1-\alpha)\epsilon^{\alpha}(1+\alpha\epsilon^{\alpha-1}\bar{\Phi}^{-1})\bar{\Phi}^{\frac{\beta}{1-\alpha}} \\
&= (1-\theta)(1-\alpha)\epsilon^{\alpha}\bar{\Phi}^{\frac{\beta}{1-\alpha}}N_s\left(\mu+\frac{1-\mu}{1+n^F}+\frac{1-\mu}{1+n^F}\alpha\epsilon^{\alpha-1}\bar{\Phi}^{-1}\right).
\end{aligned}
$$

Now, excess supply of good 2 by country F (Exs^F_2) is total production of good 2 less total domestic consumption of that good. Thus, $Exs^F_2 = N^F_s x^*_{2s,F} - (N^F_s c^*_{2ys,F} + N^F_{s-1}c^*_{2os,F})$. Hence,

$$
\begin{aligned}
Exs^F_2 = (1-\alpha)\epsilon^{\alpha}\bar{\Phi}^{\frac{\beta}{1-\alpha}}N^F_s\Bigg\{&-\frac{\alpha}{\beta-\alpha}-(1-\theta)\left(\mu+\frac{1-\mu}{1+n^F}\right) \\
&+ \frac{1-\mu}{1+n^F}\alpha\epsilon^{\alpha-1}\bar{\Phi}^{-1}\left(\frac{1-\alpha}{\beta-\alpha}\right)\Bigg\}. \qquad (77)
\end{aligned}
$$

Thus, $Exs^F_2 > 0$ if

$$
-\frac{\alpha}{\beta-\alpha}-(1-\theta)\left(\mu+\frac{1-\mu}{1+n^F}\right)+\frac{1-\mu}{1+n^F}\alpha\epsilon^{\alpha-1}\bar{\Phi}^{-1}\left(\frac{1-\alpha}{\beta-\alpha}\right) > 0.
$$

We have

$$
\bar{\Phi}^{-1} = \frac{1}{\alpha\epsilon^{\alpha-1}[(1-\alpha)-(1-\theta)(\beta-\alpha)]}\left\{\bar{N}\frac{\mu(1-\theta)(\beta-\alpha)+\alpha}{(1-\mu)}+(1-\theta)(\beta-\alpha)\right\},
$$

where

$$
\bar{N} = \frac{N^F_{s+1}+N^S_{s+1}}{N^F_s+N^S_s}.
$$

Substituting the expression of $\bar{\Phi}^{-1}$ into the inequality above and rearranging terms yields

$$\left(\frac{\bar{N}}{1+n^F}-1\right)\left(\mu(1-\theta)+\frac{\alpha}{\beta-\alpha}\right)>0.$$

Since $\frac{\bar{N}}{1+n^F}-1<0$, $\mu(1-\theta)+\frac{\alpha}{\beta-\alpha}<0$. Hence, $\mu(1-\theta)<\frac{\alpha}{\alpha-\beta}$ which is always true for $\alpha>\beta$, since $\mu(1-\theta)<1$ and $\frac{\alpha}{\alpha-\beta}>1$. Therefore, $Exs_2^F>0$ for $\alpha>\beta$ and $Exs_2^F<0$ for $\alpha<\beta$.